Several characterizations of the 4–valued modal algebras

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Abstract

A. Monteiro, in 1978, defined the algebras he named tetravalent modal algebras, will be called $4-valued\ modal\ algebras$ in this work. These algebras constitute a generalization of the $3-$valued Lukasiewicz algebras defined by Moisil.

The theory of the $4-$valued modal algebras has been widely developed by I. Loureiro in [6, 7, 8, 9, 10, 11, 12] and by A. V. Figallo in [2, 3, 4, 5].

J. Font and M. Rius indicated, in the introduction to the important work [1], a brief but detailed review on the $4-$valued modal algebras.

In this work varied characterizations are presented that show the "closeness" this variety of algebras has with other well–known algebras related to the algebraic counterparts of certain logics.

1 Introduction

In 1940 G. C. Moisil [13] introduced the notion of three–valued Łukasiewicz algebra. In 1963, A.Monteiro [14] characterized these algebras as algebras $\langle A, \wedge, \vee, \sim, \nabla, 1 \rangle$ of type $(2, 2, 1, 1, 0)$ which verify the following identities:

(A1) $x \vee 1 = 1$,

(A2) $x \wedge (x \vee y) = x$,

(A3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

(A4) $\sim\sim x = x$,

(A5) $\sim(x \vee y) = \sim x \wedge \sim y$,

(A6) $\sim x \vee \nabla x = 1$,

(A7) $x \wedge \sim x = \sim x \wedge \nabla x$,

(A8) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

L.Monteiro [15] proved that A1 follows from A2, $\cdots$, A8, and that A2, $\cdots$, A8, are independent.

From A2, $\cdots$, A5 it follows that $\langle A, \wedge, \vee, \sim, 1 \rangle$ is a De Morgan algebra with last element 1 and first element $0 = \sim 1$.

In 1969 J. Varlet [16] characterized three–valued Łukasiewicz algebras by means of other operations. Let $\langle A, \wedge, \vee, *, +, 0, 1 \rangle$ be an algebra of type $(2, 2, 1, 1, 0, 0)$ where $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bound distributive lattice with least element 0, greatest element 1 and the following properties are satisfied:

(V1) $x \wedge x^* = 0$,

(V2) $(x \wedge y)^* = x^* \wedge y^*$,

(V3) $0^* = 1$,

(V4) $x \vee x^+ = 1$,

(V5) $(x \vee y)^+ = x^+ \wedge y^+$,

(V6) $1^+ = 0$,

(V7) if $x^* = y^*$ and $x^+ = y^+$ then $x = y$.

About these algebras he proved that it is posible to define, in the sense of [14, 15] a structure of three-valued Łukasiewicz algebra by taking $\sim x = (x \vee x^*) \wedge x^+$ and $\nabla x = x^{**}$.

Furthermore it holds $x^* = \sim \nabla x$ and $x^+ = \nabla \sim x$. Therefore three–valued Łukasiewicz are double Stone lattices which satisfy the determination principle V7. Moreover V7 may be replaced by the identity

$$(x \wedge x^+) \wedge (y \vee y^*) = x \wedge x^+.$$

Later, in 1978, A. Monteiro [14] considered the 4–valued modal algebras $\langle A, \wedge, \vee, \sim, \nabla, 1 \rangle$ of type $(2, 2, 1, 1, 0)$ which satisfy A2, $\cdots$, A7 as an abstraction of three–valued Łukasiewicz algebras.

In this paper we give several characterizations of the 4–valued modal algebras. In the first one we consider the operations $\wedge, \vee, \neg, \Gamma, 0, 1$ where $\neg x = \sim \nabla x$, $\Gamma x = \nabla \sim x$ are called strong and weak negation respectively.

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2 A characterization of the 4–valued modal algebras

Theorem 2.1 *Let $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1 \rangle$ be an algebra of type $(2, 2, 1, 1, 0, 0)$ where $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice with least element 0, greatest element 1 and the operators $\triangledown, \sim$ are defined on A by means of the formulas:*

(D1) $\triangledown x = \neg \neg x,$

(D2) $\sim x = (x \vee \neg x) \wedge \Gamma x.$

Then $\langle A, \wedge, \vee, \sim, \triangledown, 1 \rangle$ is a 4–valued modal algebra if and only if it satisfies the following properties:

(B1) $x \wedge \neg x = 0,$

(B2) $x \vee \Gamma x = 1,$

(B3) $\neg x \wedge \Gamma \neg x = 0,$

(B4) $\Gamma x \vee \neg \Gamma x = 1,$

(B5) $\Gamma (x \wedge y) = \Gamma x \vee \Gamma y,$

(B6) $\neg (x \vee y) = \neg x \wedge \neg y,$

(B7) $\neg (x \wedge \neg y) = \neg x \vee \neg \neg y,$

(B8) $\Gamma (x \vee \Gamma y) = \Gamma x \wedge \Gamma \Gamma y,$

(B9) $(x \vee y) \wedge \Gamma (x \vee y) \leq x \vee \neg x,$

(B10) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma (x \vee y),$

where $a \leq b$ if and only if $a \wedge b = a$ or $a \vee b = b$. Moreover, $\neg x$ and Γx denote $\sim \triangledown x$, and $\triangledown \sim x$ respectively.

The verification of the neccesary condition does not offer any special difficulty; therefore we omit the proof. For the sufficient condition we need the following lemmas and corollaries:

Lemma 2.1 *If $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1 \rangle$ is an algebra of type $(2, 2, 1, 1, 0, 0)$ which verifies the properties B1, $\cdots$, B10 of theorem 2.1 then it holds:*

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(B11) $\ulcorner 0 = 1$,

(B12) $\neg 1 = 0$,

(B13) $\neg x \leq \ulcorner x$,

(B14) $\neg 0 = 1$,

(B15) $\ulcorner 1 = 0$,

(B16) $\ulcorner x \wedge \ulcorner \ulcorner x = 0$,

(B17) $\neg x \vee \neg \neg x = 1$,

(B18) $\neg \neg x = \ulcorner \neg x$,

(B19) $\ulcorner \ulcorner x = \neg \ulcorner x$,

(B20) $\neg x \wedge \ulcorner \ulcorner x = 0$,

(B21) $x \leq \neg \neg x$,

(B22) $\ulcorner \ulcorner x \leq x$,

(B23) $\neg \neg \neg x = \neg x$,

(B24) $\ulcorner \ulcorner \ulcorner x = \ulcorner x$,

(B25) $\neg \ulcorner x \leq x$,

(B26) $\ulcorner \ulcorner \neg x = \neg x$,

(B27) $\ulcorner \ulcorner \ulcorner \neg x = \neg \neg x$,

(B28) $\ulcorner ((x \vee \neg x) \wedge \ulcorner x) = \neg \neg x$,

(B29) $\neg \neg \ulcorner x = \ulcorner x$,

(B30) $\neg \neg \neg \ulcorner x = \ulcorner \ulcorner x$,

(B31) $\neg ((x \wedge \ulcorner x) \vee \neg x) = \ulcorner \ulcorner x$,

(B32) $\ulcorner \neg \neg x = \neg x$.

Proof. We only check B18, B22, B28 and B31.

(B18) Then $\ulcorner \neg x \leq \neg \neg x$ and by B13 $\neg \neg x \leq \ulcorner \neg x$.

(B22) $x = x \vee \Gamma\, 1,$ [B15]

$\qquad = x \vee \Gamma\,(x \vee \Gamma\, x),$ [B2]

$\qquad = x \vee (\Gamma\, x \wedge \Gamma\,\Gamma\, x),$ [B8]

$\qquad = x \vee \Gamma\,\Gamma\, x.$ [B2]

(B28) $\Gamma\,((x \vee \neg x) \wedge \Gamma\, x) = \Gamma\,(x \vee \neg x) \vee \Gamma\,\Gamma\, x,$ [B5]

$\qquad\qquad = \Gamma\,(x \vee \Gamma\,\Gamma\,\neg x) \vee \Gamma\,\Gamma\, x,$ [B26]

$\qquad\qquad = (\Gamma\, x \wedge \neg\neg x) \vee \Gamma\,\Gamma\, x,$ [B8,B27]

$\qquad\qquad = \neg\neg x.$ [B2,B21,B22]

(B31) $\neg((x \wedge \Gamma\, x) \vee \neg x) = \neg(x \wedge \Gamma\, x) \wedge \neg\neg x,$ [B6]

$\qquad\qquad = \neg(x \wedge \neg\neg\Gamma\, x) \wedge \neg\neg x,$ [B29]

$\qquad\qquad = (\neg x \vee \Gamma\,\Gamma\, x) \wedge \neg\neg x,$ [B7,B30]

$\qquad\qquad = \Gamma\,\Gamma\, x.$ [B1,B21,B22]

$\square$

Corollary 2.1 (Axiom A4) $\sim\sim x = x.$

Proof. First, we observe that from B13 and D2 we obtain

(D3) $\sim x = (x \wedge \Gamma\, x) \vee \neg x.$

Then
$$\sim\sim x = (((x \wedge \Gamma\, x) \vee \neg x) \wedge \Gamma\,((x \wedge \Gamma\, x) \vee \neg x)) \vee \neg((x \wedge \Gamma\, x) \vee \neg x), \quad \text{[D3]}$$
$$= (((x \wedge \Gamma\, x) \vee \neg x) \vee \neg\neg x) \vee \Gamma\,\Gamma\, x, \qquad \text{[B28,B31,D3,D2]}$$
$$= ((x \wedge \Gamma\, x \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)) \vee \Gamma\,\Gamma\, x,$$
$$= (x \wedge \Gamma\, x \wedge \neg\neg x) \vee \Gamma\,\Gamma\, x, \qquad \text{[B1]}$$
$$= x. \qquad \text{[B22,B2,B21,B22]}$$
$\square$

Corollary 2.2 (Axiom A6) $\sim x \vee \nabla x = 1.$

Proof.
$$\sim x \vee \nabla x = ((x \vee \neg x) \wedge \Gamma\, x) \vee \neg\neg x, \qquad \text{[D2,D1]}$$
$$= (x \wedge \Gamma\, x) \vee \neg x \vee \neg\neg x, \qquad \text{[D3]}$$
$$= (x \vee \Gamma\, x) \vee 1 = 1. \qquad \text{[B17]}$$
$\square$

Corollary 2.3 (Axiom A7) $x \wedge \sim x = \sim x \wedge \nabla x$.

Proof.

$$
\begin{aligned}
\sim x \wedge \nabla x &= ((x \wedge \Gamma\, x) \vee \neg x) \wedge \neg\neg x, & \text{[D3,D1]}\\
&= (x \wedge \Gamma\, x \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x), &\\
&= x \wedge \Gamma\, x = (x \wedge \Gamma\, x) \vee 0, & \text{[B21,B1]}\\
&= (x \wedge \Gamma\, x) \vee (x \wedge \neg x), & \text{[B1]}\\
&= ((x \wedge \Gamma\, x) \vee x) \wedge ((x \wedge \Gamma\, x) \vee \neg x), &\\
&= x \wedge \sim x. & \text{[D3]}
\end{aligned}
$$

$\square$

Lemma 2.2 *The following properties hold:*

(B33) *if* $x \le y$ *then* $\neg y \le \neg x$ *and* $\Gamma\, y \le \Gamma\, x$,

(B34) $\sim\Gamma\, x = \Gamma\,\Gamma\, x$,

(B35) $\sim(\neg x \wedge \Gamma\, y) = \neg\neg x \vee \Gamma\,\Gamma\, y$,

(B36) $\Gamma\,\Gamma\, (y \vee \neg\neg x) = \Gamma\,\Gamma\, y \vee \neg\neg x$,

(B37) $\Gamma\,\Gamma\, (x \vee y) = \neg\neg x \vee \Gamma\,\Gamma\, y$,

(B38) *if* $x \le y$ *then* $\sim y \le \sim x$,

(B39) $\neg x \wedge \Gamma\, y \le \Gamma\, (x \vee y)$,

(B40) $\neg x \wedge \sim y \le (x \vee y) \vee \neg (x \vee y)$,

(B41) $x \wedge \Gamma\, x \wedge \sim y \le \Gamma\, (x \vee y)$,

(B42) $\neg x \wedge \sim y \le \Gamma\, (x \vee y)$.

Proof.

We check only B34, B35, B36, B38, B39, B40 and B41.

$$
\begin{aligned}
\text{(B34)} \quad \sim\Gamma\, x &= \Gamma\,\Gamma\, x \wedge (\Gamma\, x \vee \neg\Gamma\, x), & \text{[D2]}\\
&= \Gamma\,\Gamma\, x \wedge (\Gamma\, x \vee \Gamma\,\Gamma\, x), & \text{[B19]}\\
&= \Gamma\,\Gamma\, x, & \text{[B2]}
\end{aligned}
$$

(B35) (1) $\sim(\neg x \wedge \Gamma y) = \Gamma\,(\neg x \wedge \Gamma y) \wedge ((\neg x \wedge \Gamma y) \vee \neg\,(\neg x \wedge \Gamma y))\,.$ [D2]

On the other hand (2) $\Gamma\,(\neg x \wedge \Gamma y) = \Gamma\,\neg x \vee \Gamma\,\Gamma\,y,$ [B5]

$$= \neg\neg x \vee \Gamma\,\Gamma\,y,$$ [B18]

and

(3) $\neg\,(\neg x \wedge \Gamma y) = \neg\Gamma\,y \vee \neg\neg x,$ [B7]

$$= \neg\neg x \vee \Gamma\,\Gamma\,y,$$ [B19]

Then B35 follows from (1), (2) and (3).

(B36) $\Gamma\,\Gamma\,y \vee \neg\neg x = \Gamma\,\Gamma\,y \vee \Gamma\,\Gamma\,\neg\neg x,$ [B26]

$$= \Gamma\,(\Gamma\,y \wedge \Gamma\,\neg\neg x),$$ [B5]

$$= \Gamma\,(\Gamma\,y \wedge \neg x),$$ [B32]

$$= \Gamma\,(\Gamma\,y \wedge \Gamma\,\Gamma\,\neg y),$$ [B26]

$$= \Gamma\,\Gamma\,(y \vee \Gamma\,\neg x),$$ [B8]

$$= \Gamma\,\Gamma\,(y \vee \neg\neg x),$$ [B18]

(B38) Let x, y be such that

(1) $x \leq y.$
Then

(2) $\sim y \vee \sim x = (y \wedge \Gamma\,y) \vee \neg y \vee \neg x \vee (x \wedge \Gamma\,x),$ [D3]

$$= (y \wedge \Gamma\,y) \vee \neg x \vee (x \wedge \Gamma\,x),$$ [(1),B34]

$$= \neg x \vee ((y \vee x) \wedge (y \vee \Gamma\,x) \wedge (\Gamma\,y \vee x) \wedge (\Gamma\,y \vee \Gamma\,x)),$$

$$= \neg x \vee (y \wedge (y \vee \Gamma\,x) \wedge (\Gamma\,y \vee x) \wedge \Gamma\,x).$$ ((1),B34)

Furthermore

(3) $1 = x \vee \Gamma\,x,$ [B2]

$$\leq y \vee \Gamma\,x,$$ [(1)]

Then

(4) $\sim y \vee \sim x = \neg x \vee (y \wedge (\Gamma\,y \vee x) \wedge \Gamma\,x),$ [(2),(3)]

$$= \neg x \vee (\Gamma\,x \wedge ((y \wedge \Gamma\,y) \vee (y \vee x))),$$

$$= \Gamma\,x \wedge (\neg x \vee (y \wedge y) \vee x),$$ [(1),B13]

(5) $y \wedge \Gamma\,y \leq x \vee \neg x.$ [(1),B9]

Then

$$\sim y \vee \sim x = \Gamma\,x \wedge (\neg x \vee x),$$ [(4),(5)]

$$= \sim x,$$ [D2]

(B39) From B34 and B35 we have

$\sim\Gamma\,(x\vee y)=\Gamma\,\Gamma\,(x\vee y),$

$\sim(\neg x\wedge y)=\neg\neg x\vee\Gamma\,\Gamma\,y,$

and by B37 it results

(1) $\sim\Gamma\,(x\vee y)\leq\sim(\neg x\vee\Gamma\,y)$

From (1), B38 and corollary 2.3. $\neg x\wedge\Gamma\,y\leq\Gamma\,(x\wedge y).$

(B40) $\neg x\wedge\sim y\wedge((x\vee y)\vee\neg(x\vee y))=\sim y\wedge((\neg x\wedge(x\vee y))\vee(\neg x\wedge\neg(x\vee y))),$

$$=\sim y\wedge((\neg x\wedge y)\vee(\neg x\wedge\neg y)),$$

[B1,B33]

$$=\sim y\wedge\neg x\wedge(y\vee\neg y),$$
$$=\neg x\wedge\sim y.\qquad\text{[D3]}$$

(B41) (1) $x\wedge\Gamma\,x\wedge\sim y=x\wedge\Gamma\,x\wedge\Gamma\,y\wedge(y\vee\neg y),\qquad\text{[D3]}$

$$=(x\wedge\Gamma\,x\wedge\Gamma\,y\wedge y)\vee(x\wedge\Gamma\,x\wedge\Gamma\,y\wedge\neg y).$$

On the other hand

(2) $x\wedge\Gamma\,x\wedge y\wedge\Gamma\,y\leq\Gamma\,(x\vee y),\qquad\text{[B10]}$

(3) $\qquad\quad\Gamma\,x\wedge\neg y\leq\Gamma\,(x\vee y).\qquad\text{[B39]}$

Then

$x\wedge\Gamma\,x\wedge\sim y\leq\Gamma\,(x\vee y)\vee(x\wedge\Gamma\,(x\vee y))=\Gamma\,(x\vee y).\;\;[(1)(2)(3)]$

□

Corollary 2.4 (Axiom A5) $\sim(x\vee y)=\sim x\wedge\sim y$

Proof.

We have

$\sim(x\vee y)=\sim x\wedge\sim y,\qquad\qquad\qquad\qquad\qquad\text{[B38]}$

On the other hand

(1) $\sim x\wedge\sim y=(x\wedge\Gamma\,x\wedge\sim y)\vee(\neg x\wedge\sim y),\qquad\qquad\text{[D2]}$

(2) $x\wedge\Gamma\,x\wedge\sim y\leq(x\vee y)\vee\neg(x\vee y),$

(3) $x\wedge\Gamma\,x\wedge\sim y\leq\sim(x\vee y)\,,\qquad\qquad\qquad\text{[(3),(B41),(D2)]}$

(4) $\neg x\wedge\sim y\leq\sim(x\vee y)\,,\qquad\qquad\qquad\text{[(B40),(B42),(D2)]}$

(5) $\sim x\wedge\sim y\leq\sim(x\vee y).\qquad\qquad\qquad\qquad\text{[(1),(2),(4)]}$

Finally, taking into account that $(A,\wedge,\vee,0,1)$ is a bounded distributive lattice with least element 0, greatest element 1, the sufficient condition of theorem 2.1. follows from corollaries 2.3, 2.7, 2.4 and 2.5. □

3 Other characterizations

The following characterization of 4–valued modal algebras is easier than that given in theorem 2.1.

Theorem 3.1 *Let* $(A, \wedge, \vee, \curlyvee, \neg, 1)$ *be an algebra of type* (2,2,1,1,0) *where* $(A, \wedge, \vee, \sim, 1)$ *is a De Morgan algebra with last element* 1 *and first element* $0 = \sim 1$. *If* ∇ *is an unary operation defined on* A *by means of the formula* $\nabla x = \sim \neg x$. *Then* A *is a 4–valued modal algebra if and only if it verifies:*

(T1) $x \wedge \neg x = 0$.

(T2) $x \vee \neg x = x \vee \sim x$.

    Furthermore $\neg x = \sim \nabla x$.

Proof.
    We check only sufficient condition

$$\text{(A6)} \quad \sim x \vee \nabla x = \sim x \vee \sim \neg x = \sim(x \wedge \neg x) = 1. \qquad\qquad [\text{T1}]$$

$$\text{(A7)} \quad \sim x \wedge \nabla x = \sim x \wedge \sim \neg x, \qquad\qquad\qquad\qquad\qquad [\text{T2}]$$
$$= \sim(x \vee \neg x),$$
$$= \sim(x \vee \sim x),$$
$$= x \wedge \sim x.$$

<div align="right">□</div>

Remark 3.1 *In a 4–valued modal algebra the operation considered in* 2.1, *generally does not coincide with the pseudo-complement* * *as we can verify in the following example:*
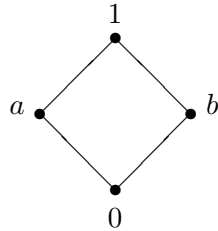

x	$\sim x$	∇x
0	1	0
a	a	1
b	b	1
1	0	1

figure 1   table 1

    we have

x	$\neg x$	x^*
0	1	1
a	0	b
b	0	a
1	0	0

table 2

However all finite 4–valued modal algebra is a distributive lattice pseudo complemented. We do not know whether this situation holds in the non-finite case. This suggests that we consider a particular class of De Morgan algebras.

Definition 3.1 *An algebra* $(A, \wedge, \vee, \sim, ^*, 1)$ *of type* (2,2,1,1,0) *is a modal De Morgan p–algebra if the reduct* $(A, \wedge, \vee, \sim, 1)$ *is a De Morgan algebra with last element 1 and first element* $0 = \sim 1$, *the reduct is a pseudo–complemented meet–lattice and the following condition is verified*

$$\text{H1) } x \vee \sim x \leq x \vee x^*$$

Example 3.1 *The De Morgan algebra whose Hasse diagram is given in figure 2 and the operations* $\sim$ *and* * *are defined in* table 3
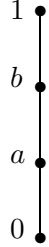

x	$\sim x$	x^*
0	1	1
a	b	0
b	a	0
1	0	0

figure 2   table 3

is not a modal De Morgan p–algebra because $b = (a \vee \sim a) \nleq a \vee a^* = a$.

Theorem 3.2 *If we define on a modal De Morgan p–algebra* $\langle A, \wedge, \vee, \sim, ^*, 1 \rangle$ *the operation* $\neg$ *by means of the formula* $\neg x = x^* \wedge \sim x$ *then the algebra* $\langle A, \wedge, \vee, \neg, 1 \rangle$ *verifies the identities* T1 *and* T2.

Proof.

(T1) $x \wedge \neg x = x \wedge x^* \wedge \sim x = 0 \wedge \sim x = 0$.

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(T2) $x \vee \neg x = x \vee (x^* \wedge \sim x) = (x \vee x^*) \wedge (x \vee \sim x),$

$$= (x \vee \sim x). \hspace{4cm} \text{[H1]}$$

$\square$

Remark 3.2 *By [4] we know that every finite modal 4–valued algebra A is direct product of copies of* T2*,* T3 *and* T4*, where* T2={0,1} *and* T3={0,a,1} *are modal De Morgan p–algebra we conclude that A is also a modal De Morgan p–algebra.*

We do not know whether this situation holds in the non-finite case.

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